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THESIS

SENSITIVITY ANALYSIS OF OPTIMAL SOLUTIONS
FOR
OBJECTIVE FUNCTIONS WITH STOCHASTIC PARAMETERS

by

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March 1977

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FOR
OBJECTIVE FUNCTIONS WITH STOCHASTIC PARAMETERS

by

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ABSTRACT

The purpose of this paper is to develop a definitional framework for the constrained optimization of an objective function with stochastic parameters. Included in this definitional framework are the construction of a deterministic reformulation for the stochastic problem, guidelines for selecting an algorithm for optimizing the deterministic reformulation and a criterion, called invariance, which is used for solution sensitivity analysis in the context of stochastic parameter behavior. Illustration of the application of the definitional framework and an analysis of the conditions imposed by the invariance criterion is provided for linear, piecewise linear, and quadratic functions.

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I. INTRODUCTION

The overhaul of a naval vessel is a three phase event; a planning phase of approximately one year, an industrial work phase, and a post-overhaul and inspection phase. The typical overhaul has a budget in the millions of dollars and involves several thousand individual jobs which are normally aggregated into a much smaller number of managerial groups. In this paper these managerial groups will be referred to as stages.

A major event in the overhaul planning phase is the job screening conference. The purpose of this conference is to identify the stages in which advance planning and procurement should commence. This is accomplished by individually reviewing the proposed jobs in each of the stages and selecting, within stage and overall budget limits, those jobs which will be accomplished during the overhaul. This screening process is, in effect, an implicit approach toward an optimal allocation of the available resources. As these resources are increasingly limited there is growing interest in formalizing the screening process into an explicit optimization which uses the standard tools of mathematical programming. One formidable obstacle in the development of the explicit optimization model is the identification of a measure of effectiveness which is applicable to each of the stages. Even assuming the existence of some measure there remains a basic problem which arises from the method of determining return as a function of resource allocation for each stage. These stage return functions will rarely be explicitly known and therefore must be estimated from available data. Regardless

of the quantity and quality of the data the estimation will result in stage return functions with parameters subject to random error. Thus, in addition to identification of a measure of effectiveness, a prerequisite of explicit optimization of overhaul planning allocations is an ability to conduct constrained optimization of an objective function whose parameters are subject to random error.

The optimization of an objective function with parameters subject to random error falls into the general realm of stochastic programming. As noted by Nemhauser [8], the concept of optimization in the presence of stochastic behavior is not well defined, and is the subject of continuing discussion. From this discussion two definitional frameworks have arisen. One is highly theoretical and is exemplified by statistical decision and game theory. The other framework is more computationally oriented and is exemplified by the method which addresses the creation of a deterministic reformulation of the stochastic program to which the optimization can be applied. There are several variants but the central idea of a deterministic reformulation is to select a particular value from the range of each of the stochastic parameters, a common choice is the expectation, and to treat these values as deterministic parameters for the purpose of optimization. The approach is open to considerable criticism, the most common being the failure to account for possible effects on the solution caused by differences between the realizations of the random variables and the values used in the optimization.

The goal of this research is to develop, for an arbitrary measure of effectiveness, a definitional framework for the optimization of the general class of stochastic programs posed by the overhaul planning problem. The framework is to be computationally oriented and based on a deterministic reformulation. The method defines a concept

called "invariance" which is used to assess the effect of stochastic parameter behavior on the set of stages in which no advance planning and procurement actions are indicated. The criterion is directed toward this set rather than the set of stages in which advance planning and procurement should commence since the costs associated with the erroneous inclusion of a stage in the no-planning-and-procurement set are much larger than the costs associated with the erroneous inclusion of a stage in the advance-planning-and-procurement set.

Chapter two will develop a mathematical model of the overhaul planning problem, define the invariance criterion and develop the conditions imposed by this criterion on the deterministic reformulation and its optimization. Based on these conditions a solution approach for the optimization of the overhaul planning problem is proposed and its application illustrated in chapters three, four, and five for linear, piecewise linear, and quadratic stage return functions.

II. PROBLEM FORMULATION AND CONCEPT DEVELOPMENT

A. MATHEMATICAL FORMULATION

Assuming that there are n stages to be considered by the screening conference a mathematical formulation of the overhaul planning problem is dependent on the identification of a composition operator used to form the overall objective function from the individual stage return functions. As the overhaul planning problem is of an economic nature, that is: it addresses the allocation of scarce resources, any arbitrary measure of effectiveness is, in effect, a form of utility measure. A reasonable choice for the composition of the stage returns into an aggregate measure of the utility of the total overhaul package is summation. That is, stage returns are assumed additive. Based on these assumptions one mathematical formulation of the overhaul planning problem is:

$$\begin{aligned} & \text{MAX} \quad \sum_{j=1}^n r_j^*(d_j) \\ & \text{s/t} \quad \sum_{j=1}^n d_j \leq BD \\ & \quad 0 \leq d_j \leq ub_j \quad j=1,2,\dots,n \end{aligned}$$

where

$r_j^*(d_j)$ is the j^{th} stage stochastic return function.

d_j is the j^{th} stage decision variable.
 BD is the total budget constraint.
 ub_j is the j^{th} stage budget constraint.

Based on the economic context of the problem, it is assumed that stage allocations are subject to diminishing returns, thus the stage return functions are concave. In the same context it is also reasonable to assert that the sum of the individual stage budget constraints exceeds the total budget constraint, thus at optimality the total budget constraint will be binding.

B. DEFINITIONAL FRAMEWORK

1. Deterministic Stage Returns

As previously noted the concept of optimization in the presence of stochastic behavior is not well defined. The stochastic optimization problem formulated above is to be defined in terms of a deterministic reformulation, each $r_j^*(d_j)$ being converted to a deterministic $r_j(d_j)$ by selecting from the respective ranges a specific value for each of the stochastic parameters. The optimization of the deterministic reformulation is to be achieved by application of one of the existing algorithms of mathematical programming. The specific algorithm will depend on the form of the stage return functions used in the deterministic reformulation.

2. Solution Sensitivity Analysis

The effect of stochastic parameter behavior on the optimal solution of the deterministic reformulation can be observed from three viewpoints, in terms of the values of the decision variables and the value of the objective function, in terms of the solution set or in terms of the complement of the solution set.

The solution set S^+ for an n -variable m -constraint problem is defined as the set of variables different from zero.

The complement of the solution set, denoted S^0 , is the set of variables which are zero.

As this problem is in a planning context the solution set S^+ , which identifies the stages in which advance planning and procurement should commence, and its complement S^0 , which identifies the stages in which there should be no advance planning and procurement, are more important than the values of the decision variables or the value of the objective function. Since the costs associated with a stage being incorrectly included in S^0 are much larger than the

cost of a stage being incorrectly included in S^+ , the sensitivity analysis of the solution to the deterministic reformulation should be directed toward the complement of the solution set. Invariance, defined below, is the criteria which will be used to assess the sensitivity of the complement of the solution set to the stochastic parameter behavior.

The optimal solution of the deterministic reformulation is invariant with respect to the specified ranges of the stochastic parameters, if within those specified ranges no zero valued variable is required to depart the complement of the solution set, that is become positive, to maintain optimality.

Basically for a solution set to be invariant the variables of the deterministic reformulation which have an optimal value of zero must remain at zero for parametric perturbations throughout the specified ranges of the stochastic parameters. There are two concepts in the definition of invariance which require additional development.

a. Specified Parameter Range

Invariance is defined in terms of specified ranges for each of the stochastic parameters. Some of these parameters may be the result of a statistical regression and as such they are assumed to be normally distributed random variables with a nominal range of $(-\infty, \infty)$. Ranges of this magnitude would always violate the invariance criterion,

thus one requirement for meaningful analysis of solution sensitivity is range restriction for each of the stochastic parameters. The restriction must be accomplished about the value of the parameter used in the deterministic reformulation and must provide a reasonable chance for any realization of the random variable to be in the restricted range. Additional assumptions for this paper are that the restricted ranges are symmetric about the parameter value used in the deterministic reformulation and that perturbations in any given restricted range have no effect on perturbations in any other restricted range.

The individual restricted ranges may be developed with statistical confidence intervals, [4] and [9], from previous experience, from known physical limitations or any other approach which yields a restricted range meeting the guidelines established above. The specific method used to establish restricted ranges for an invariance analysis is not dealt with in this paper.

The restricted range for a stochastic parameter c_j^* will be denoted as;

$$c_j^* = c_j + \phi_j$$

$$-\phi_j^L \leq \phi_j \leq \phi_j^L$$

where

c_j is used in the deterministic reformulation.

ϕ_j is a random variable.

ϕ_j^L is the limit for perturbations to c_j^* .

b. Changes to the Solution Set

A requirement of the invariance criterion is an ability to identify when a variable must depart the complement of the solution set to maintain optimality. For an algorithm to be useful for the optimization of the deterministic reformulation in the context of the invariance criterion it must be possible to perturb a problem parameter without necessitating a complete reapplication of the algorithm.

C. SOLUTION APPROACH

For arbitrary return functions the stochastic program posed by the overhaul planning problem can be optimized within the definitional framework of invariance by:

1. selecting the parameter values to be used in the deterministic reformulation,
2. restricting the range of each stochastic parameter about the value selected in step one,
3. selecting an appropriate algorithm, depending on the form of the deterministic reformulation return functions, and optimizing the deterministic reformulation,
4. introducing perturbations within the established ranges to the stochastic parameters and checking the invariance condition.

This approach will be illustrated for common classes of stage return functions and for each class tests of invariance will be identified. The classes of return function to be examined are linear, piecewise linear, and quadratic.

III. LINEAR STAGE RETURN FUNCTIONS

The simplest function in the general class of concave functions is the linear function and while the assumption of linear stage returns may not be very realistic in the economic context it does provide a useful starting point for illustrating the application of the definitional framework developed to optimize overhaul planning allocations.

Assume that each of the n stage return functions is a linear function of the form

$$r_j^*(d_j) = a_j^* + c_j^* d_j.$$

The initial steps in the proposed solution approach are to select the values of a_j^* and c_j^* , respectively a_j and c_j , for use in the deterministic reformulation and to establish restricted ranges about a_j and c_j . Given the estimates of a_j and c_j with appropriate restricted ranges about them and the general mathematical formulation of the overhaul planning problem the linear deterministic reformulation is

$$\text{MAX} \sum_{j=1}^n (a_j + c_j d_j)$$

$$\text{s/t} \sum_{j=1}^n d_j \leq BD$$

$$0 \leq b_j \leq ub_j \quad j=1,2,\dots,n.$$

As a_j^* affects only the value of the objective function it can be dropped from the formulation with no loss of generality. The resultant simplified formulation is

$$\text{MAX} \sum_{j=1}^n c_j d_j$$

$$\text{s/t} \sum_{j=1}^n d_j \leq BD$$

$$0 \leq d_j \leq ub_j \quad j=1,2,\dots,n$$

and the invariance condition need only be checked for the restricted ranges of c_j^*

$$c_j^* = c_j + \phi_j$$

$$-\phi_j^L \leq \phi_j \leq \phi_j^L.$$

The deterministic reformulation above is a linear program and the most common algorithm for such an

optimization is simplex. The simplex algorithm can accept parameter perturbations via the parametric sensitivity analysis techniques outlined in Gass [5] and is therefore useful for the optimization of the linear deterministic reformulation in the context of an invariance analysis.

As a reference for the investigation of the conditions imposed by the invariance criterion, consider the following four stage (n=4) example problem.

$$\begin{aligned}
 &\text{MAX } 1.4d_1 + 1.6d_2 + 1.3d_3 + 1.7d_4 \\
 &\text{s/t } d_1 + d_2 + d_3 + d_4 \leq 10 \\
 &\quad d_1 \leq 5 \\
 &\quad \quad d_2 \leq 3 \\
 &\quad \quad \quad d_3 \leq 4 \\
 &\quad \quad \quad \quad d_4 \leq 4 \\
 &\quad \quad \quad \quad \quad d_j \geq 0 \quad j=1,2,3,4.
 \end{aligned}$$

For this problem the optimal simplex tableau is

B	c _B	1	2	3	4	5	6	7	8	9	d
1	1.4	1	0	1	0	1	0	-1	0	-1	3
6	0	0	0	-1	0	-1	1	1	0	1	2
2	1.6	0	1	0	0	0	0	1	0	0	3
8	0	0	0	1	0	0	0	0	1	0	4
4	1.7	0	0	0	1	0	0	0	0	1	4
z - c _j		0	0	.1	0	1.4	0	.2	0	.3	

where columns five through nine are slack variables.

For notational simplicity and without loss of generality the following assumptions are made;

1. The standard simplex tableaus are used and are arranged so that the first n columns are the decision variables (d_j) . The $n+1^{\text{st}}$ column is the slack variable for the total budget constraint, which is binding at optimality, and columns $n+2$ through $2n+1$ are the slack variables for each of the decision variables.
2. At optimality there is no degeneracy and no alternate optimal solution.

Notationally let

B be the optimal solution set or basis.

C_B be the vector of the c_j for $j \in B$.

P_j be the j^{th} column of the optimal tableau.

I_B be the vector of the ϕ_j for $j \in B$.

C_B^* be the vector sum of C_B and I_B .

It is important to note that for the slack variables both c_j and ϕ_j are zero.

Linear optimizations have a property that for any variable to depart the complement of the solution set a variable must depart the solution set. Given this one-to-one correspondence between changes in the solution set and its complement the requirements of the invariance

criterion can be satisfied if within the specified restricted ranges for the stochastic parameters there are no changes to the solution set. In the simplex algorithm changes to the solution set can occur only if the algorithm optimality criteria are violated. Therefore, the key to an invariance analysis in the linear case is the optimality criterion of the simplex algorithm. The optimality criterion of the simplex algorithm is that

$$(z_j - c_j) = c_B P_j - c_j \geq 0 \quad \forall j.$$

Introducing the perturbations allowed by the restricted parameter ranges has the following effect;

$$\begin{aligned} (z_j - c_j)^* &= c_B^* P_j^* - c_j^* \\ &= (C_B + I_B) P_j - (c_j + \phi_j) \\ &= (C_B P_j - c_j) + (I_B P_j - \phi_j) \end{aligned}$$

or

$$(z_j - c_j)^* = (z_j - c_j) + (I_B P_j - \phi_j)$$

and as the perturbations were introduced at optimality the necessary condition to maintain optimality is that

$$-(I_B P_j - \phi_j) < (z_j - c_j) \quad \forall j.$$

For the solution to be invariant the optimality maintenance condition must be satisfied throughout the restricted ranges of the stochastic parameters, that is for

$$-\phi_j^L \leq \phi_j \leq \phi_j^L.$$

The presence of the I_B term in the optimality maintenance condition means that for any column j the maintenance of

optimality is dependent on perturbations to parameter c_j and on perturbations to the parameters of the variables in the optimal basis. Thus in the optimality maintenance condition for any column j there are mutual dependence effects from the stochastic parameter perturbations.

The constraint matrix of the problem has a structure which produces an optimal tableau whose elements are either minus one, plus one, or zero. Further the assumption that the total budget constraint is binding at optimality produces an optimal tableau with only four types of columns. These column types are identified as follows;

Type O = columns of basic variables.

Type A = columns of non-basic decision variables.

Type B = columns of non-basic slack variable.

Type C = the column of the non-basic slack variable of the total budget constraint.

These special features reduce the mutual dependence effects of parameter perturbations in the optimality maintenance condition and allow the development of tests of invariance for each of the column types identified.

A. TESTS OF INVARIANCE

Due to the special structure of the problem the simplex algorithm will bring the decision variables into the basis at their upper bounds ub_j sequentially in decreasing order of the rates of return. Under these circumstances optimality will occur when the entering decision variable

causes the total budget constraint to become binding. The decision variable which causes the total budget constraint to become binding will be the final variable to enter the basis and will be denoted d_f . For the example problem d_f is d_1 .

1. Column Type C

As shown in Gass [5] parametric perturbations produce no change in the optimality conditions for type 0 columns, that is

$$(z_j - c_j)^* = (z_j - c_j) = 0.$$

Thus there is no requirement to examine the columns of the optimal basis under the invariance criterion. In the example problem columns one, two, four, six, and eight are type 0 columns.

2. Column Type A

Column three of the example problem is a type A column. In general any column j of type A has three non-zero elements. These elements are located as follows:

A plus one in the row of d_f . In the example problem this is row one.

A plus one in the row of the slack variable associated with this, the j^{th} , variable.

A minus one in the row of the slack variable associated with d_f . From the tableau assumptions the slack variable associated with d_f is variable $n+f+1$. In the example problem this is variable six which is associated with tableau row two.

As the ϕ_j of the slack variables are zero the scalar product $I_{Bj} P_j$ for a type A column must always be

$$I_{Bj} P_j = \phi_f$$

and the optimality condition is

$$-(\phi_f - \phi_j) < (z_j - c_j).$$

As $(z_j - c_j)$ is a positive quantity the worst condition for invariance is at

$$\phi_f = -\phi_f^L$$

$$\phi_j = \phi_j^L$$

and the test for invariance in a type A column is

$$\phi_f^L + \phi_j^L < (z_j - c_j).$$

3. Column Type B

Examples of type B columns can be found in columns seven and nine of the example problem. In general any column j of type B has three non-zero elements. These elements are located as follows:

A minus one in the row of the d_f . In the example problem this is row one.

A one in the row of the slack variable of d_f . From the tableau assumptions the slack variable associated with d_f is variable $n+f+1$. In the example problem this is variable six which is associated with tableau row two.

A one in the row of the decision variable associated with this slack variable, by the tableau assumptions this is variable $j-(n+1)$.

For this type of column the scalar product $I_B P_j$ is

$$I_B P_j = -\phi_f + \phi_{j-(n+1)}$$

and the optimality condition is

$$-(-\phi_f + \phi_{j-(n+1)}) < (z_j - c_j).$$

The worst condition for invariance occurs when

$$\begin{aligned}\phi_f &= \phi_f^L \\ \phi_{j-(n+1)} &= -\phi_{j-(n+1)}^L\end{aligned}$$

and the test of invariance for a type B column is

$$\phi_f^L + \phi_{j-(n+1)}^L < (z_j - c_j).$$

4. Column Type C

An example of a type C column is column five of the example problem. In general for column $(n+1)$, which is the only type C column, there are two non-zero elements. These non-zero elements are located as follows:

A one in the row of d_f . in the example problem this is row one.

A minus one in the row of the slack variable associated with d_f . From the tableau assumptions the slack variable associated with d_f is variable $n+f+1$. In the example problem this is variable six which is associated with tableau row two.

The scalar product $I P_{B \ n+1}$ for the type C column is

$$I P_{B \ n+1} = \phi_f$$

and the optimality condition is

$$-\phi_f < (z_j - c_j).$$

The worst condition for invariance occurs when

$$\phi_1^L = -\phi_f^L$$

and the test for invariance in the type C column is

$$\phi_f^L < (z_j - c_j).$$

The solution set is said to be invariant with respect to the specified parameter ranges if the tests of invariance are not violated for any of the columns of the optimal tableau.

The tests developed above are sufficient but not necessary conditions for invariance. Under certain conditions violation of the sufficiency test for a type B column may not result in a violation of invariance. This will occur when a variable in S^+ can become zero without requiring a variable in S^0 to become positive. The condition for this occurrence is that it must be possible to allocate all of the units of resource from the variable which is dropping to zero to the variable which is in the solution set but not at its upper bound (d_f). If the violation of the sufficiency test for a type B column occurs in column i a new condition for invariance should be checked. This condition is

$$ub_{i-(n+1)} > ub_f - d_f.$$

The solution will not be invariant if this condition is satisfied since the resource not allocated to d_f must be

allocated to some variable in S^0 to maintain optimality. If the condition is not satisfied the solution may be invariant but to determine whether or not the solution is invariant requires additional tableau pivots.

Recognizing that the simplex algorithm brings the decision variables into the solution set at their upper bounds sequentially in decreasing order of their rates of return and that optimality occurs when a decision variable

can enter only at some value less than its upper bound a logical statement of a test of invariance is

$$\max_{j \in S^0} c_j^* < \min_{j \in S^+} c_j^*.$$

This condition is equivalent to the tests of invariance developed above.

IV. PIECEWISE LINEAR RETURN FUNCTIONS

Since piecewise linear functions can exhibit diminishing marginal returns they are economically more realistic than the simple linear function. This increase in economic realism is gained at the loss of some of the mathematical simplicity of the simple linear return model. However, the highly desirable property of linearity is retained. Piecewise linear functions generally arise from two sources, one is segmented linear regression. The second and more common source is the approximation of non-linear functions.

Assume that the individual stage return functions are piecewise linear functions of the form

$$r_j(d_j) = a_j^* + c_j^1 d_j^1 + c_j^2 d_j^2 + \dots + c_j^k d_j^k$$

where

$$d_j = d_j^1 + d_j^2 + \dots + d_j^k$$

$$0 \leq d_j^m \leq ub_j^m \quad m=1,2,\dots,k$$

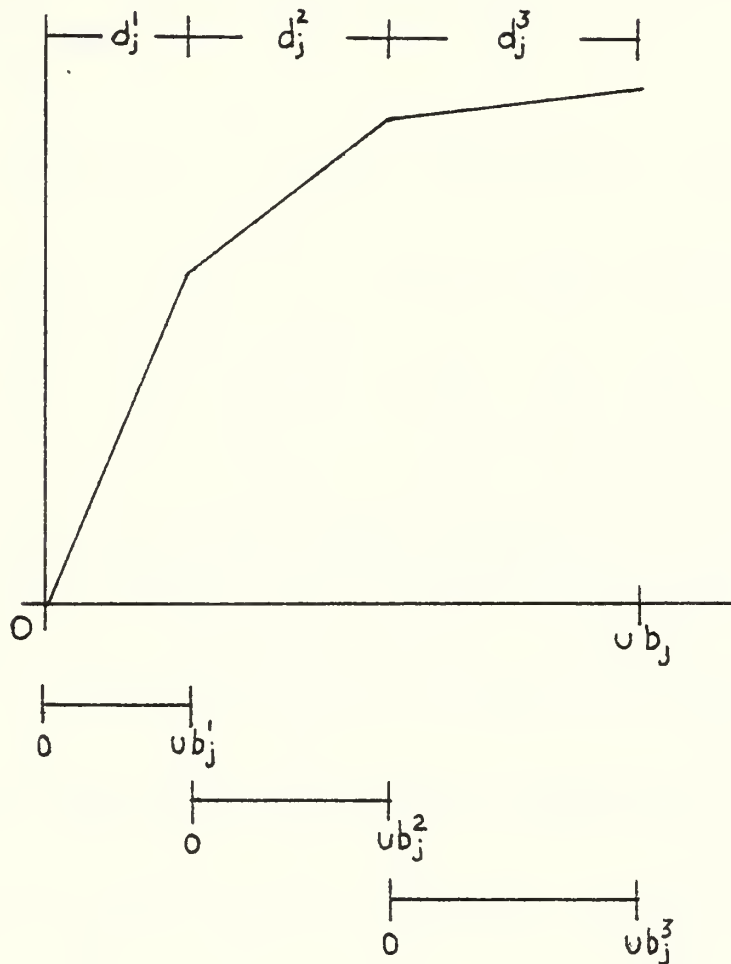
and to ensure the desired concavity

$$b_j^m > b_j^{m+1} \quad m=1,2,\dots,k-1.$$

Given the economic context of the problem it is reasonable to assert that marginal returns are strictly positive. This requires the piecewise linear function to be monotone and is ensured if

$$b_j^k > 0.$$

Graphically a typical three segment piecewise linear stage return function will have the form



A further requirement which ensures that the algorithm used for the optimization of the deterministic reformulation correctly models the piecewise linear function is that d_j^{m+1} remain at zero until d_j^m is at ub_j^m . It should also be noted that

$$\sum_{m=1}^k ub_j^m = ub_j \quad \forall j,$$

thus the original stage constraints remain in force. The assumption of k segments for each of the piecewise linear functions is for notational simplicity and is made without loss of generality.

The parameters c_j^{m*} can be regarded as approximations to the slope of the j^{th} stage return function over the interval $[ub_j^{m-1}, ub_j^m]$ and as such they are subject to random error. It must be noted that in addition to the guidelines for restricted ranges established in chapter two the restricted ranges in the piecewise linear case must ensure that the function remains concave and monotone increasing. If the limits for a restricted range are $-\phi_j^{mL}$ and ϕ_j^{mL} the maintenance of concavity requires that

$$c_j^m - \phi_j^{mL} > c_j^{m+1} + \phi_j^{(m+1)L} \quad m=1,2,\dots,k-1$$

and to maintain the monotonic behavior requires that

$$c_j^k - \phi_j^{kL} > 0.$$

Given the estimates of the c_j^m 's with appropriate restricted ranges about them and the general mathematical formulation of the overhaul planning problem the piecewise

linear deterministic reformulation is

$$\begin{aligned}
 & \text{MAX} \sum_{j=1}^n (a_j + \sum_{m=1}^k c_j^m d_j^m) \\
 & \text{s/t} \sum_{j=1}^n \sum_{m=1}^k d_j^m \leq BD \\
 & 0 \leq d_j^m \leq ub_j^m \quad j=1,2,\dots,n \\
 & \quad \quad \quad m=1,2,\dots,k.
 \end{aligned}$$

As a_j^* affects only the value of the objective function it can be dropped from the formulation with no loss of generality. The simplified formulation which results is

$$\begin{aligned}
 & \text{MAX} \sum_{j=1}^n \sum_{m=1}^k c_j^m d_j^m \\
 & \text{s/t} \sum_{j=1}^n \sum_{m=1}^k d_j^m \leq BD \\
 & 0 \leq d_j^m \leq ub_j^m \quad j=1,2,\dots,n \\
 & \quad \quad \quad m=1,2,\dots,k
 \end{aligned}$$

and the invariance criterion need only be checked for the restricted ranges of c_j^{m*}

$$c_j^{m*} = c_j^m + \phi_j^m$$

$$-\phi_j^{mL} \leq \phi_j^m \leq \phi_j^{mL}.$$

The deterministic reformulation above is a linear program except for the requirement that d_j^{m+1} remain at zero until d_j^m is at ub_j^m . Fortunately this requirement does not prevent the use of the simplex algorithm since, as shown by Dantzig [1] and [2], the concavity of the objective function ensures that the requirement will be satisfied at optimality. Thus the simplex algorithm is applicable for the optimization of the piecewise linear deterministic reformulation.

For notational simplicity and without loss of generality the following assumptions are made;

1. The standard simplex tableaus are used and are arranged so that the first nk columns are the decision variables (d_j^m) . The $(nk+1)^{st}$ column is the slack variable for the total budget constraint, which is binding at optimality, and columns $nk+2$ through $2nk+1$ are the slack variables for each of the decision variables.
2. The column arrangement is by block according to the variable index j and within each block the k segments are in order from one to k . Thus the column arrangement is of the form

$$d_1^1 \ d_1^2 \ \dots \ d_1^k \ d_2^1 \ \dots \ d_n^k \ d_{n+1}^1 \ d_{n+2}^1 \ \dots \ d_{2n+1}^k.$$

3. There is a tableau row for each variable to enforce its respective bound. Thus the tableau will have $nk+1$ rows and $2nk+1$ columns.
4. At optimality there is no degeneracy and no alternate optimal solution.

Notationally let

B be the optimal solution set or basis.

C_B be the vector of the b_j^m for $d_j^m \in B$.

P_j^m be a column of the optimal tableau.

I_B be the vector of ϕ_j^m for $d_j^m \in B$.

C_B^* be the vector sum of C_B and I_B .

The optimization of piecewise linear functions is a linear optimization and as such there is a one-to-one correspondence between changes to the basic and non-basic variables. However in the piecewise linear case this is not equivalent to a one-to-one correspondence between changes to the solution set and its complement. This is due to the fact that in the piecewise linear case invariance is directed at the original variables d_j not at the piecewise segment

variables d_j^m . The invariance condition can be stated as;

if within the specified restricted ranges for the stochastic parameters the d_j^1 which are in S^0 remain at zero the

solution is invariant. This condition can be checked through the simplex algorithm optimality criteria. In the notation of the piecewise linear case the optimality criterion of the simplex algorithm is

$$(z_j^m - c_j^m) = C_B^m P_j^m - c_j^m \geq 0 \quad \forall j, m=1,2,\dots,k.$$

Introducing the perturbations allowed by the restricted parameter ranges has the following effect;

$$\begin{aligned} (z_j^m - c_j^m)^* &= C_B^{*m} P_j^m - c_j^{*m} \\ &= (C_B + I_B) P_j^m - (c_j^m + \phi_j^m) \\ &= (C_B P_j^m - c_j^m) + (I_B P_j^m - \phi_j^m) \end{aligned}$$

or

$$(z_j^m - c_j^m)^* = (z_j^m - c_j^m) + (I_B P_j^m - \phi_j^m)$$

and to maintain optimality

$$-(I_B P_j^m - \phi_j^m) < (z_j^m - c_j^m).$$

The invariance criterion requires that the optimality maintenance condition be satisfied throughout the specified restricted ranges of the parameters, that is for

$$-\phi_j^{mL} \leq \phi_j^m \leq \phi_j^{mL}.$$

The piecewise linear optimality maintenance condition is, with notational differences, the same as the simple linear case. The only real differences are that there are now nk total decision variables instead of the n for the simple linear case and that the optimality maintenance condition need not be checked for all of the columns of the

optimal tableau for the solution to be invariant. The columns for which the optimality maintenance condition need not be checked are those of the non-basic structural variables which do not affect invariance, that is the columns of d_j^m where $m > 1$. These variables do not affect invariance since the invariance condition for the piecewise linear problem is stated in terms of the d_j^1 which are in S^0 . The special features of the problem, namely the constraint matrix structure and the assumption of a binding total budget constraint, are intact and the optimal tableau has the same four column types and element arrangements as the simple linear tableau. Thus the invariance analysis for piecewise linear functions is virtually identical to that of the simple linear case. The difference is that the only type A columns which must be checked for invariance in the piecewise linear case are those associated with d_j^1 's which are in S^0 . Since the analysis for the columns which must be checked is the same in the piecewise linear problem as in the simple linear problem the tests of the simple linear case will simply be restated in the current notation for those columns of the piecewise linear case which are relevant under the invariance criterion.

A. TESTS OF INVARIANCE

As in the simple linear case the subscript f denotes the decision variable which upon entry into the basis causes the total budget constraint to become binding.

1. Column Type C (basic variables)

No examination of the columns of the optimal basis is required under the invariance criterion.

2. Column Type A (non-basic structural variables)

As pointed out above the only columns of type A which are relevant under the invariance criterion are those associated with d_j^1 's. The test of invariance for a relevant type A column is

$$\phi_f^L + \phi_j^{1L} < (z_j^1 - c_j^1).$$

3. Column Type B (non-basic slack variables)

The test for invariance for any column of type B is

$$\phi_f^L + \phi_{j-(n+1)}^{mL} < (z_j^m - c_j^m).$$

4. Column Type C (budget constraint slack variable)

The test for invariance for the type C column is

$$\phi_f^L < (z_{nk+1} - c_{nk+1}).$$

The solution set is said to be invariant with respect to the specified parameter ranges if the tests of invariance are satisfied for all relevant columns of the optimal tableau.

As in the simple linear case the tests above are sufficient but not necessary conditions for invariance. Under certain conditions a violation of the type B column sufficiency test may not be a violation of invariance. This occurs when a variable in S^+ can become zero without requiring a variable in S^0 to become positive. The condition for this occurrence is that it must be possible to allocate all of the units of resource from the variable which is dropping to zero to the variable which is in the solution set but not at its upper bound (d_f). However in the piecewise linear case it is not possible to state a condition which checks for this behavior. This is due to the fact that violation of the allocation condition stated above may not result in a violation of invariance since the units of resource not allocated to d_f may be allocated either to a d_j^m where $m > 1$, which would not violate invariance, or to a d_j^1 which would violate invariance.

As in the simple linear case the simplex algorithm will bring the decision variables of the piecewise linear case into the solution set at their upper bounds sequentially in decreasing order of their rates of return. Optimality will still be achieved when an entering decision variable encounters the total budget constraint and the logical statement of the tests of invariance remains

$$\max_{j \in S^0} c_j^{1*} < \min_{j \in S^+} c_j^{m*} \quad m=1,2,\dots,k.$$

The solution is said to be invariant if this condition is satisfied throughout the specified restricted ranges.

V. QUADRATIC STAGE RETURN FUNCTIONS

Quadratic functions are another move toward economic realism for the overhaul planning problem stage return functions. The advantage of a quadratic function is that marginal returns can be made to be continuously decreasing whereas the piecewise linear function could only incrementally decrease marginal return. The gain in realism with the quadratic function has a large price, the loss of linearity. However this loss is partially offset since the quadratic function is one of the simplest of the non-linear functions and there is some advantage over piecewise linear functions because of the reduced number of parameters and variables.

Assume that each of the n stage return functions is a quadratic function of the form

$$r_j^*(d_j) = a_j^* d_j - \frac{1}{2} b_j^* d_j^2.$$

The conditions for the desired properties of decreasing but always positive marginal returns, or monotonic concavity, are that the first derivative be non-negative and the second derivative be strictly negative. This means that

$$\begin{aligned} a_j &> b_j \text{ ub}_j \\ b_j &> 0. \end{aligned}$$

It should be noted that the restricted ranges in the quadratic case must, in addition to satisfying the guidelines of chapter two, maintain the monotonic concavity

of the individual stage return functions. If the limits of the restricted ranges of a_j^* and b_j^* are respectively $\pm\phi_j^L$ and $\pm\theta_j^L$ the maintenance of monotonic concavity requires that

$$b_j - \theta_j^L > 0$$

and

$$a_j - \phi_j^L > (b_j + \theta_j^L)ub_j.$$

Given the estimates of a_j and b_j with appropriate restricted ranges about them and the general mathematical formulation of the overhaul planning problem the quadratic deterministic reformulation is

$$\begin{aligned} \text{MAX} \quad & \sum_{j=1}^n a_j d_j - \frac{1}{2} b_j d_j^2 \\ \text{s/t} \quad & \sum_{j=1}^n d_j \leq BD \\ & 0 \leq d_j \leq ub_j \quad j=1,2,\dots,n. \end{aligned}$$

The invariance condition must be checked for the restricted ranges of b_j^*

$$b_j^* = b_j + \theta_j$$

$$-\theta_j^L \leq \theta_j \leq \theta_j^L$$

and for the restricted ranges of a_j^*

$$a_j^* = a_j + \phi_j$$

$$-\phi_j^L \leq \phi_j \leq \phi_j^L.$$

It will be notationally convenient to shift to matrix notation where

D is an (nx1) vector of decision variables d_j .

B is an (nx1) vector of parameters b_j .

A is an (nxn) positive definite diagonal matrix of parameters a_j .

I is an (nxn) identity matrix.

1^* is an (nx1) vector of 1's.

UB is an (nx1) vector of the stage constraints ub_j .

The deterministic reformulation can be written in matrix notation as

$$\text{MAX } R(D) = A^t D - \frac{1}{2} B^t D B$$

$$\text{s/t } D^t 1^* \leq B D$$

$$0 \leq I D \leq U B.$$

This is a convex non-linear program and the Kuhn-Tucker conditions are necessary and sufficient for optimality.

The initial attempt to optimize the deterministic reformulation and apply the invariance criterion involved the use of quadratic programming, specifically the algorithm of Dantzig as described by Boot [3]. While this algorithm is simplex based and technically satisfies the requirements for the application of the invariance criterion it involves a reformulation via the Kuhn-Tucker conditions and the resultant problem proved to be conceptually and computationally difficult. The specific difficulties were;

1. The reformulated problem is conceptually an existence problem rather than an optimization problem. Thus there is no objective function in the usual sense and this prevents the determination of optimality maintenance conditions.
2. The parameters to which perturbations are applied are in the constraint matrix and on the right hand side of the reformulated problem. This situation is almost impossible to handle when perturbations are applied to more than one parameter.

The second algorithm selected for the optimization was the reduced gradient algorithm, [6], [7], and [10]. This algorithm is very similiar to the simplex procedure, identifying variables as either basic or non-basic. One of the important differences between this algorithm and simplex is that here the basic variables must be positive and the non-basic variables may be different from zero. The reduced gradient algorithm can also handle parameter perturbations without having to reapply the entire algorithm, thus the algorithm is applicable under the requirements of the invariance criterion. To put the deterministic equivalent into the standard format of the reduced gradient algrithm add slack variables to the constraint matrix and denote the

result as

$$W = \left[\begin{array}{ccc|c} t & * & 1 & 0' \\ D & 1 & 1 & 0' \\ \hline & & & \\ ID & 0 & I & \end{array} \right] \leq \left[\begin{array}{c} BD \\ \hline UB \end{array} \right]$$

where

$0'$ is an $(1 \times n)$ vector of zeros.

0^* is an $(n \times 1)$ vector of zeros.

additionally at optimality let

Z be the set of the $n+1$ basic variables.

N be the set of n non-basic variables.

P be the matrix of the columns of W associated with the basic variables.

C be the matrix of the columns of W associated with the non-basic variables.

$q_{k,1}$ be the 1th element in the k th row of $Q = P^{-1}C$.

It should be noted that

1. The solution set S^+ has elements from both Z and N .
2. The complement of the solution set S^0 has elements only from N .
3. The column arrangement of W is the same as the column arrangement of the constraint matrix in the linear case.
4. For the slack variables ϕ_j , θ_j , and d_j are all zero.

In the quadratic case there is no one-to-one correspondence between changes in the solution set and its complement. Thus the use of the invariance criterion in the quadratic case requires a direct application of the definition of invariance, that is the invariance analysis must be focused on the complement of the solution set, S^0 .

The maintenance of optimality in the quadratic case is not necessary for invariance, in fact optimality will probably never be maintained. However, the optimality condition of the reduced gradient algorithm does provide a method for checking the invariance criterion. The optimality condition of the reduced gradient algorithm is that all elements u_j of the movement vector U , defined below, be equal to zero. The elements of the movement vector are defined as

$$u_j = \begin{cases} 0 & \text{if } RG_j \leq 0 \text{ and } d_j = 0 \\ RG_j & \text{otherwise} \end{cases} \quad \forall j \in N$$

where RG_j is the j^{th} component of the reduced gradient RG and

$$RG = -\nabla_Z R(D) P^{-1} C + \nabla_N R(D)$$

or

$$RG_j = - \sum_{z \in Z} (a_z - b_z d_z) q_{z,j} + (a_j - b_j d_j).$$

The key for the invariance criterion is that the variables in the complement of the solution set (S^0) are specifically required to have a reduced gradient component which is non-positive. Departures from S^0 can only occur if the perturbations within the specified restricted ranges cause the reduced gradient component of one of the variables in S^0 to become positive, thus invariance is dependent on the changes to the reduced gradient components of the variables in S^0 produced by the parametric perturbations. Introducing the parameter perturbations allowed by the restricted ranges has the following effect on the reduced gradient.

$$\begin{aligned}
 RG_j^* &= - \sum_{z \in Z} (a_z^* - b_z^* d_z) q_{z,j} + a_j^* - b_j^* d_j \\
 &= - \sum_{z \in Z} (a_z + \phi_z - (b_z + \theta_z) d_z) q_{z,j} \\
 &\quad + (a_j + \phi_j) - (b_j + \theta_j) d_j \\
 &= - \sum_{z \in Z} (a_z - b_z d_z) q_{z,j} + a_j - b_j d_j \\
 &\quad - \sum_{z \in Z} (\phi_z - \theta_z d_z) q_{z,j} + \phi_j - \theta_j d_j \\
 &= RG_j - \sum_{z \in Z} (\phi_z - \theta_z d_z) q_{z,j} + \phi_j - \theta_j d_j.
 \end{aligned}$$

For invariance the RG_j^* of the variables in S^0 must remain non-positive. Noting that d_j for all variables in S^0 must be

zero the changes in the reduced gradient components of the variables in S^0 simplifies to

$$RG_j^* = RG_j - \sum_{z \in Z} (\phi_z - \theta_z d_z) q_{z,j} + \phi_j.$$

Since RG_j is a non-positive quantity the condition for the maintenance of non-positive RG_j^* can be expressed as

$$RG_j \geq - \sum_{z \in Z} (\phi_z - \theta_z d_z) q_{z,j} + \phi_j \quad \forall j \in S^0$$

and for invariance this condition must be maintained for all variables in S^0 and throughout the specified restricted ranges, that is for

$$\begin{aligned} -\phi_j^L &\leq \phi_j \leq \phi_j^L \\ -\theta_j^L &\leq \theta_j \leq \theta_j^L. \end{aligned}$$

The condition for the maintenance of non-positive RG_j^* for the variables in S^0 can be simplified by examining the structure of the constraint matrix W , which has exactly the same structure as the constraint matrix of the linear case. Although in the quadratic case there can be more than one variable in the solution that is not at its upper bound the structure of W allows for the occurrence of only one of them in the basis set Z . This arises as follows;

The occurrence of a variable which is not at its upper bound in the basis set Z requires that both the variable and its slack be in Z . This requires that the $(n+1) \times (n+1)$ matrix P , which must be non-singular,

contain both the column of the structural variable and the column of the slack. Each variable/slack pair after the first fills two column dimensions but only one row dimension in P . If an attempt is made to identify $i > 1$ variables as not being at their upper bounds there will be $(n - i + 1)$ filled column dimensions but only $(n - i)$ filled row dimensions in P . Of the remaining $(n - i + 1)$ variables in the basis, each is identified as being at its upper bound and fills only one row and only one column dimension in P . Thus the net result of trying to identify more than one variable as not being at its upper bound is an unfilled row dimension in P and this row of zeros makes P singular, which is in violation of the requirement for P .

Since only one variable can be identified as being in the solution but not at its upper bound the basis in the quadratic case is exactly the same as in the linear case and the representation of the non-basic columns of W , that is Q , at optimality is exactly the same as in the linear case. This means that the columns of Q associated with the variables in S^0 have the same structure as the columns of the linear case which are associated with non-basic, and therefore zero valued, variables. The columns of Q associated with variables in S^0 are either type A, B, or C columns where

type A = columns associated with structural variables.

type B = columns associated with slack variables.

type C = the column associated with the slack of the total budget constraint.

A. TESTS OF INVARIANCE

Denote the set of variables which are in the solution but not at their upper bounds as the set F . The variable of F which is identified as being in the basis but not at its upper bound will be denoted as d_f . With this notation the arrangement of the elements in the quadratic type A, B, and C columns is identical to the arrangement of the elements in the linear type A, B, and C columns. For details of these arrangements see pages 22, 23, and 25 respectively.

1. Column Type A (structural variables)

For $d_j \in S^0$ that are structural variables the columns of Q will have the structure of the type A column, see page 22. Under this condition the summation in the RG_j^* maintenance condition becomes

$$-(\phi_f - \theta_{ff} d_f)$$

and the condition for the maintenance of non-positive RG_j^* is

$$|RG_j| > -(\phi_f - \theta_{ff} d_f) + \phi_j.$$

To obtain a test of invariance the variable identified as being in the solution but not at its upper bound must be the variable in F which has the largest deviation from its optimal value marginal return, that is select d_f to

$$\max_{f \in F} (\phi_f^L + \theta_f^L d_f).$$

The test of invariance for a type A column is

$$|RG_j| > \max_{f \in F} (\phi_f^L + \theta_f^L d_{\hat{f}}) + \phi_j^L.$$

2. Column Type B (slack variables)

For $d_j \in S^0$ that are slack variables the columns of Q will have the structure of the type B column, see page 23. Under this condition the summation term in the RG_j^* maintenance condition becomes

$$-(\phi_f - \theta_f d_f) - (\phi_{j-(n+1)} - \theta_{j-(n+1)} d_{j-(n+1)}).$$

The test of invariance for type B columns is

$$\begin{aligned} |RG_j| &> \max_{f \in F} (\phi_f^L + \theta_f^L d_f) \\ &\quad + (\phi_{j-(n+1)}^L - \theta_{j-(n+1)}^L d_{j-(n+1)}) + \phi_j^L. \end{aligned}$$

3. Column Type C (budget constraint slack variable)

The slack variable associated with the total budget constraint is variable $n+1$. The column of Q associated with d_{n+1} is a type C column, see page 25. Under this condition

the summation term in the RG_j^* maintenance condition becomes

$$-(\phi_f - \theta_f d_f)$$

and the test of invariance is

$$|RG_j| > \max_{f \in F} (\phi_f^L + \theta_f^L d_f)$$

The solution is said to be invariant with respect to the specified parameter ranges if the tests of invariance hold for all $d_j \in S^0$.

The tests developed above are sufficient but not necessary conditions for invariance. Under certain conditions violation of the sufficiency test for a type B column may not result in a violation of invariance. This will occur when a variable in S^+ can become zero without requiring a variable in S^0 to become positive. The condition for this occurrence is that it must be possible to allocate all of the units of resource from the variable dropping to zero to the variables in the solution set that are not at their upper bounds. If the sufficiency test for a type B column is violated for slack variable i a new condition for invariance should be checked. This condition is

$$d_{i-(n+1)} > \sum_{f \in F} (ub_f - d_f).$$

If this condition is satisfied the solution is not invariant since the units of resource not allocated to the variables in F must be allocated to a variable in S^0 to maintain optimality. If this condition is violated the solution may be invariant and to determine whether or not the solution is invariant requires additional algorithm iterations.

As in the linear case there is a logical statement of the tests of invariance. This is derived by recognizing that the reduced gradient algorithm will allocate incremental units of resource in decreasing order of the maximum marginal returns. At optimality all variables in the solution but not at their upper bounds will have a common marginal rate of return Δ . As long as the parametric perturbations allowed by the restricted ranges do not cause the marginal return at zero of a variable in S^0 to become larger than the marginal return of any variable in S^+ the solution will be invariant. This test of invariance is equivalent to the tests developed above and can be expressed as

$$\max_{j \in S^0} (a_j + \phi_j^L) < \min_{j \in S^+} (a_j + \phi_j^L) - (b_j + \theta_j^L) d_j$$

If within the specified restricted ranges the condition is satisfied the solution is said to be invariant.

VI. CONCLUSIONS AND EXTENSIONS

The invariance criterion and the definitional framework in which it is applied provide an approach toward the optimization of the stochastic program posed by the overhaul planning problem. The complement of the optimal solution set of the deterministic reformulation identifies the stages in which no advance planning and procurement should be undertaken and the solution sensitivity analysis under the invariance criterion provides some control over the risk involved in basing decisions on a deterministic solution to a stochastic problem. The key is that the risk is only controlled since even if a solution satisfies the invariance criterion there is a positive probability that the realizations of the stochastic parameters will fall outside the restricted ranges used for the invariance criterion. Thus even if a solution satisfies the invariance criterion it should not be regarded as an exact answer, but as a foundation on which objective management decisions can be based. Even if a solution fails the tests of invariance there is a positive result, the stages which are susceptible to stochastic behavior can be identified. For a solution which fails to satisfy the invariance criterion there are two ways to move toward invariance.

1. Reduce the variance of the estimate used in the deterministic reformulation. This generally requires an enlargement of the data base used in the estimation.
2. Reduce the restricted ranges on which the tests of invariance were based. This will increase the risk

associated with decisions based on the solution as there will be a greater probability of realizations of the stochastic parameters falling outside the restricted range.

A significant improvement in the invariance criterion could be achieved if the level of risk associated with an invariant solution could be specified. This would require the development of a probability statement addressing the joint probability of violating the specified restricted ranges of the stochastic parameters. Unfortunately a realistic development of this probability statement would require dropping the assumption of independence for the parametric perturbations and would necessitate the development of joint restricted ranges for the dependent parameters. This will greatly complicate the mathematics of the invariance analysis.

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